Analytic Extraction of IR/UV divergences in multiloop diagram

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High1-2014 KIAS-NCTS Joint workshop

2014/2/14

Outline

- Introduction
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  2. UV, IR divergences
  3. Parameterization for multiloop diagram
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- Summary

Motivation

Now we are in this status.

- LHC has discovered a Higgs boson which is consistent with SM prediction.
- However, it has not found any signal BSM.
- But, It should be somewhere…
- Tantalizing question is how and where to find it.

Motivation

Collider people have an answer in mind

The ways to find NP signal:

Indirect Search

Direct Search

Low energy observables.
- Top physics
- Higgs physics
- $(g-2)_\mu$, EDM
- B physics, FCNC
- EW precision

Excess in kinematic distribution
Jet observables
Somewhere else..

Resonances, bumps

→ Entering Into Precision era

Motivation

In order for that, we need to do before and after LHC14 run

- make solid understanding of theory prediction
  - Higher order corrections in collider observables.
  - Resummation of large logs for given process.
  - Better understanding about theory uncertainty

- pin down fundamental parameters
  - $\alpha_s(M_Z)$,
  - m_t, m_W, m_Z,
  - m_h, y_t, λ_{hhh}, λ_{hhhh}

→ *We easily meet multiloop calculation*

Motivation

The aim of this work is

- Analytic separation of IR and UV divergences for given multiloop diagram.

  - Analytic computation has

  pros: Exact, Flexible, Keep analytic property

  cons: Sometimes not available, Complicated

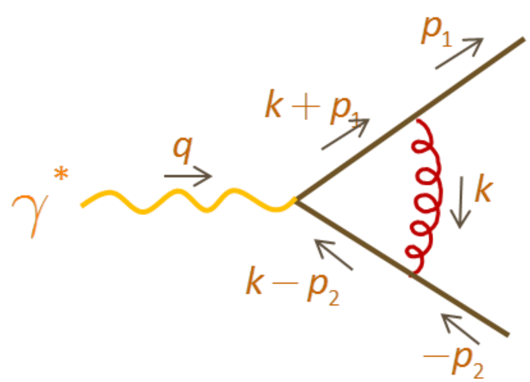
  - There are numerical methods (Sector Decomposition)
    Heinrich, 0803.4177, SecDec, FIESTA
  - There is no analytic methods in the market.

- **Automatic calculation with computing code.**

  "Revolution of NLO" – MC@NLO, POWHEG, Blackhat, MCFM...
  Automatic calculation beyond NLO is still an open question.

UV, IR divergences

Let us consider the process $\gamma^* \to q\bar{q}$



Virtual correction

$$\sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(\not{k}+\not{p}_1)(\not{k}-\not{p}_2)}$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{(\not{k}+\not{p}_1)(\not{k}-\not{p}_2)}{k^2(k^2+2k\cdot p_1)(k^2-2k\cdot p_2)}$$

UV

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} \sim \frac{1}{\epsilon_{UV}}$$

IR

$k \sim 0$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 k\cdot p_1 k\cdot p_2} \sim \frac{1}{\epsilon_{IR}}$$

→ Soft divergence

IR

$k \sim p_1$

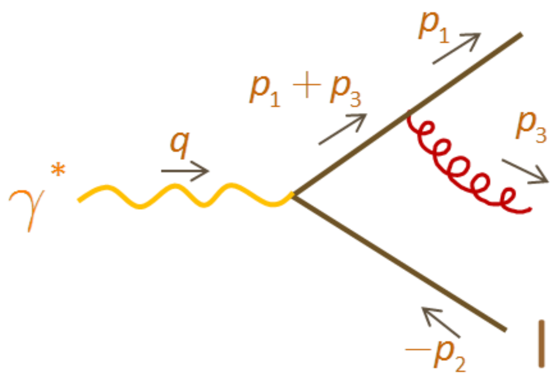
$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2+2k\cdot p_1)} \sim \frac{1}{\epsilon_{IR}}$$

→ Collinear divergence

All the UV divs. are eliminated by Renormalization.

UV, IR divergences

Let us consider the process $\gamma^* \to q\bar{q}$



Real Emission

$$\sim \int d\Pi_3 \frac{1}{2p_1 \cdot p_3}$$

IR: $p_3 \sim 0$ $\quad \sim \dfrac{1}{\epsilon_{\text{IR}}}$ → Soft divergence

IR: $p_3 \sim p_1$ $\quad \sim \dfrac{1}{\epsilon_{\text{IR}}}$ → Collinear divergence

All the IR divs. are canceled between Virtual correction and Real emission. → KLN theorem.

UV, IR divergences

Dipole subtraction method (NLO)

S. Catani, M. Seymour, (1997), S. Catani, S. Dittmaier, M. Seymour, Z. Trocsanyi (2002)

→ *Systematic method for canceling IR between virtual correction and real emission diagrams.*

$$\sigma^{\rm NLO} = \int_{m+1} \left[\left(d\sigma^R\right)_{\epsilon=0} - \left(d\sigma^A\right)_{\epsilon=0} \right] + \int_m \left[d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0}$$

Antenna subtraction (up to NNLO)

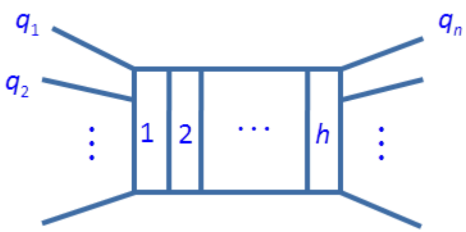
Kosower (1998), Ridder, Gehrmann, Glover (2005)

$$\begin{aligned} \mathrm{d}\sigma_{\rm NNLO} = & \int_{\mathrm{d}\Phi_{m+2}} \left(\mathrm{d}\sigma^R_{\rm NNLO} - \mathrm{d}\sigma^S_{\rm NNLO}\right) + \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\sigma^S_{\rm NNLO} + \\ & + \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^{V,1}_{\rm NNLO} - \mathrm{d}\sigma^{VS,1}_{\rm NNLO}\right) + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{VS,1}_{\rm NNLO} + \\ & + \int_{\mathrm{d}\Phi_m} \mathrm{d}\sigma^{V,2}_{\rm NNLO} , \end{aligned}$$

→ *In any case, we need to calculate IRs in virtual correction.*

Parametrization

- Feynman Parameterization



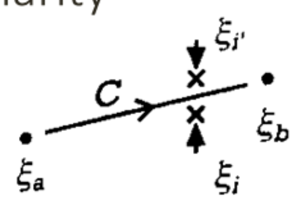
$$= \int [d^d k_1 \cdots d^d k_h] \frac{1}{\mathcal{P}_1^{a_1} \cdots \mathcal{P}_L^{a_L}} \equiv F(q_1, \cdots, q_n; d)$$

$$F_\Gamma(q_1, q_2, \cdots, q_n; d) = \int d^d k_1 \cdots d^d k_h$$

$$\times \int_0^\infty d\xi_1 \cdots \int_0^\infty d\xi_L \, \delta\left(\sum \xi_l - 1\right) \frac{\prod_l \xi_l^{a_l - 1}}{\left(\sum \mathcal{P}_l \, \xi_l\right)^a}$$

→ *Divergences arise in hyper-surface $S(k_i, \xi_i)$ that makes $\Sigma P_l \xi_l = 0$.*

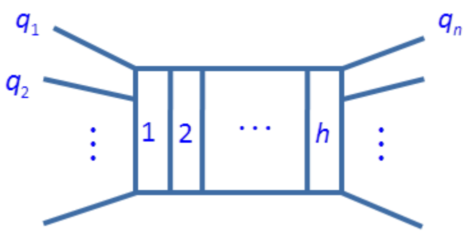
- End point singularity
- Pinch surface



G. Sterman, "Intro to QFT" (1993)

Parametrization

- alpha Parameterization



$$= \int [d^d k_1 \cdots d^d k_h] \frac{1}{\mathcal{P}_1^{a_1} \cdots \mathcal{P}_L^{a_L}} \equiv F(q_1, \cdots, q_n; d)$$

$$F\,(q_1, q_2, \cdots, q_n; d) = \frac{(-1)^a}{\prod_j \Gamma(a_j)} \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_L \delta(\sum \alpha_j - 1) \frac{\mathcal{U}^{a-(h+1)d/2} \prod_j \alpha_j^{a_j - 1}}{(-\mathcal{V} + \mathcal{U} \sum m_j^2 \alpha_j)^{a-hd/2}}$$

$\mathcal{U} = \sum_{T \in T^1} \prod_{l \notin T} \alpha_l$ → Polynomial of αs of total order h.

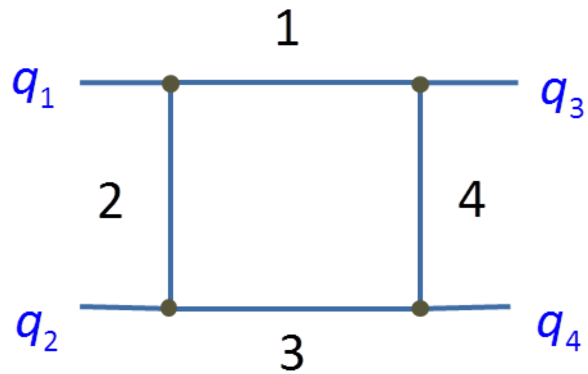
$\mathcal{V} = \sum_{T \in T^2} \prod_{l \notin T} \alpha_l (q\ \ )^2$ → Polynomial of αs of total order $h+1$.

→ *U, V are polynomials of order 1 for each variable α.*

→ *Only end-point singularity, No pinch surface.*

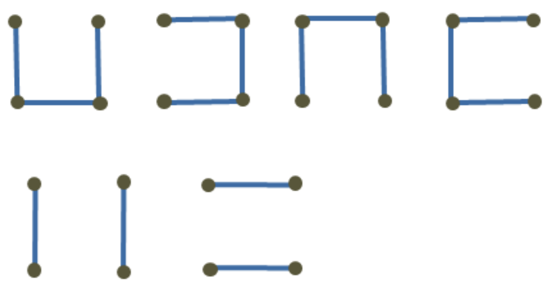
Parametrization

For example



T_1 :

T_2 :



$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$\mathcal{V} = s\,\alpha_1\alpha_3 + t\,\alpha_2\alpha_4$$

$$F(s,t;\varepsilon) = \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_4 \delta\left(\sum \alpha_j - 1\right) \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 1)^{2\varepsilon}}{\left(-t\,\alpha_1\alpha_3 - s\alpha_2\alpha_4\right)^{2+\varepsilon}}$$

Parametrization

Where are UV/IR divergences?

→ *searching for denominator to be zero.*

$$F(q_1, q_2, \cdots, q_n; d) =$$

$$\frac{(-1)^a}{\prod_j \Gamma(a_j)} \int_0^\infty d\alpha_1 \cdots d\alpha_L \frac{\delta(\sum \alpha_j - 1) \prod_j \alpha_j^{a_j - 1}}{\underbrace{\mathcal{U}^{-a+(h+1)d/2}}_{\text{UV source}} \underbrace{(-\mathcal{V} + \mathcal{U} \sum m_j^2 \alpha_j)}_{\text{IR source}}{}^{a-hd/2}}$$

Divergences arise when denominator becomes zero as same number of alphas with integer power of denominator have boundary values.

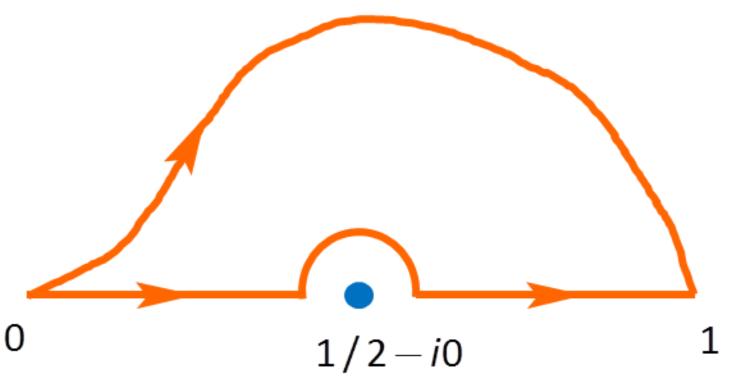
Parametrization

For example-1

$$\int_0^1 d\alpha \frac{1}{(\alpha + i0)^{1-\varepsilon}} = \frac{1}{\varepsilon}$$

$$\int_0^1 d\alpha \frac{1}{(\alpha - 1/2 + i0)^{1-\varepsilon}} = \log\left(\frac{1}{2}\right) - \log\left(-\frac{1}{2} + i0\right) = -\pi i$$

→ *zero denominator with non-boundary values of alphas generate imaginary values.*



Parametrization

For example-2

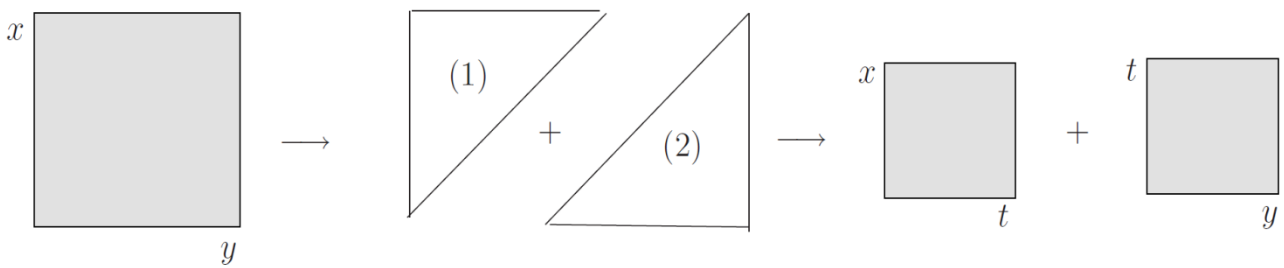
$$\int_0^1 d\alpha_1 d\alpha_2 \frac{1}{(\alpha_1+\alpha_2)^{1-\varepsilon}} = \int_0^1 d\alpha_1 \ln\left(\frac{1+\alpha_1}{\alpha_1}\right) + \mathcal{O}(\varepsilon) = 2\ln 2 + \mathcal{O}(\varepsilon)$$

→ No div. where the number of alphas with boundary values that make denominator zero is greater than integer power of denominator.

$$\int_0^1 d\alpha_1 d\alpha_2 \frac{1}{(\alpha_1+\alpha_2)^{2-\varepsilon}} = \int_0^1 d\alpha_1 \frac{\alpha_1^{\varepsilon-1} - (\alpha_1+1)^{\varepsilon-1}}{1-\varepsilon} = \frac{1}{\varepsilon} + 1 - \ln 2 + \mathcal{O}(\varepsilon)$$

Sector Decomposition

Heinrich, 0803.4177



$$\int_0^1 d\alpha_1 d\alpha_2 \frac{1}{(\alpha_1+\alpha_2)^{2-\varepsilon}} = \left(\int_{0\le\alpha_1\le\alpha_2}^1 d\alpha_1 d\alpha_2 + \int_{0\le\alpha_2\le\alpha_1}^1 d\alpha_1 d\alpha_2\right)\frac{1}{(\alpha_1+\alpha_2)^{2-\varepsilon}}$$

$$= \int_0^1 d\alpha_1 d\alpha_2 \frac{1}{\alpha_2^{1-\varepsilon}(\alpha_1+1)^{2-\varepsilon}} + \int_0^1 d\alpha_1 d\alpha_2 \frac{1}{\alpha_1^{1-\varepsilon}(\alpha_2+1)^{2-\varepsilon}}$$

$$= \frac{1}{\varepsilon} + 1 - \ln 2 + \mathcal{O}(\varepsilon)$$

→ *Perform this decomposition for the multi-dimensional hyper-surface of alphas.*

→ Algorithmic. **Suitable for numeric calculation.**

Separation of IR/UV

We first apply 'Cheng-Wu' theorem for all alphas except 1.

$$F(q_1,q_2,\cdots,q_n;d)=\frac{(-1)^a}{\prod_j \Gamma(a_j)}\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_L\,\delta\Big(\sum\alpha_j-1\Big)\frac{\mathcal{U}^{a-(h+1)d/2}\prod_j \alpha_j^{a_j-1}}{\big(-\mathcal{V}+\mathcal{U}\sum m_j^2\alpha_j\big)^{a-hd/2}}$$

We can replace $\delta(\sum\alpha_j-1)$ *with* $\delta(\sum_{\nu\in S}\alpha_\nu-1)$ *for any subset* S *of alpha variables set.*

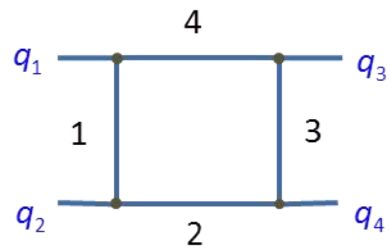
Our choice is $\delta(\sum\alpha_j-1)\rightarrow\delta(\alpha_L-1)$

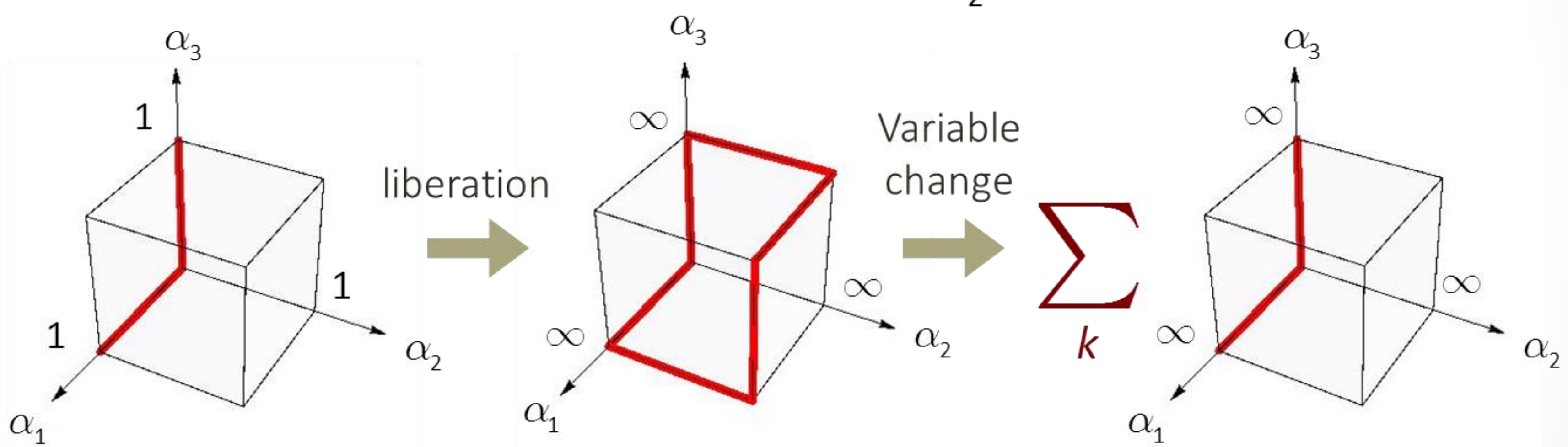
"Liberation" : we choose one alpha and make it 1, integrate from zero to infinity for all the other αs.

Cost: End point is extended from 0 to $\{0,\infty\}$

Separation of IR/UV

For example, Consider





$S_1 = \{\alpha_1, \alpha_2\}$
$S_2 = \{\alpha_2, \alpha_3\}$

$S_1 = \{\alpha_1, \alpha_2\}$
$S_2 = \{\alpha_2, \alpha_3\}$
$S_3 = \{\bar{\alpha}_1, \bar{\alpha}_2\}$
$S_4 = \{\alpha_1, \bar{\alpha}_3\}$
$S_5 = \{\bar{\alpha}_1, \alpha_3\}$
$S_6 = \{\bar{\alpha}_2, \bar{\alpha}_3\}$

$$\int_0^\infty d\alpha = \int_0^1 d\alpha + \int_1^\infty d\alpha$$

$$\alpha \to \frac{\alpha}{1+\alpha} \qquad \alpha \to \frac{1+\alpha}{\alpha}$$

Separation of IR/UV

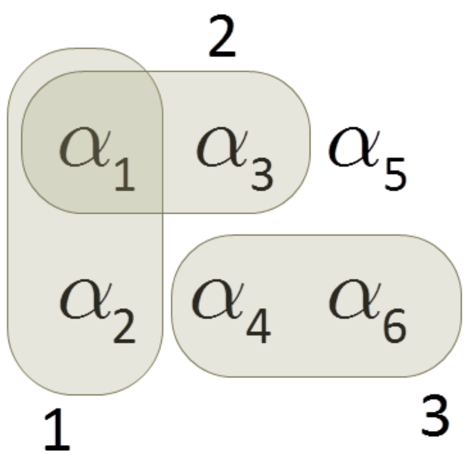
Investigate denominator to find divergences

$$\frac{1}{\mathcal{U}^{2-\varepsilon}(-\mathcal{V}+\mathcal{U}\sum m_j^2\alpha_j)^{3-\varepsilon}}$$

Construct a set S that consists of set of alphas that cause divergences

1. For example

$$S_1 = \{\alpha_1, \alpha_2\}$$
$$S_2 = \{\alpha_1, \alpha_3\}$$
$$S_3 = \{\alpha_4, \alpha_6\}$$



3. Do the variable change

$$\alpha_1 \to \eta_1\eta_2\alpha_1,$$
$$\alpha_2 \to \eta_1\alpha_2,$$
$$\alpha_3 \to \eta_2\alpha_3,$$
$$\alpha_4 \to \eta_3\alpha_4,$$
$$\alpha_5 \to \eta_3\alpha_5$$

2. Multiply by $1 = \int_0^\infty d\eta_1 d\eta_2 d\eta_3 \delta(\eta_1 - \alpha_2)\delta(\eta_2 - \alpha_3)\delta(\eta_3 - \alpha_4 - \alpha_6)$

Separation of IR/UV

The IR/UV div. are separated as

$$\int_0^{\infty} d\eta_1 d\eta_2 d\eta_3 \underbrace{\eta_1^{-1+a_1\varepsilon}\eta_2^{-1+a_2\varepsilon}\eta_3^{-1+a_3\varepsilon}}_{\text{Div. part}} \underbrace{\delta(\alpha_2-1)\delta(\alpha_3-1)\delta(\alpha_4+\alpha_6-1)F(\eta_i,\alpha_i,\varepsilon)}_{\text{Div. free, safely expanded in } \varepsilon}$$

After variable change : $\eta_j \to \dfrac{(1-\xi_j)}{\xi_j},\ \int_0^{\infty} d\eta_j \to \int_0^1 \dfrac{d\xi_j}{\xi_j^2}$

we use following expansion formula for div. part.

$$\xi^{-1+a\varepsilon} = \frac{\delta(\xi)}{a\varepsilon} + \sum_k \frac{(a\varepsilon)^k}{k!}\left[\frac{\ln^k \xi}{\xi}\right]_+$$

Analytic Computation

Remaining integrals can be done in terms of GHPLs.

GHPL : Generalized Harmonic Poly-Logarithm func.

$$G(p_1,\cdots,p_m;x) \equiv \int_0^x \frac{dy_1}{y_1-p_1}\int_0^{y_1}\frac{dy_2}{y_2-p_2}\cdots\int_0^{y_{m-1}}\frac{dy_m}{y_m-p_m}$$

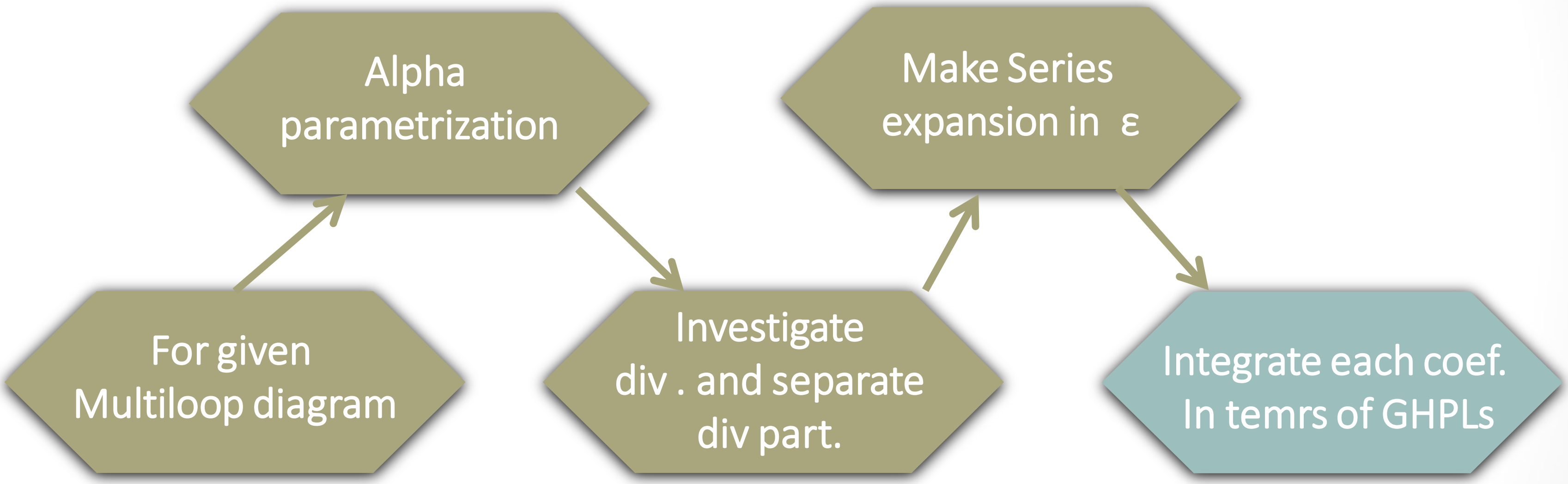
Ex: $\mathrm{Li}_n(x) = -\int_0^x \frac{dy_1}{y_1}\int_0^{y_1}\frac{dy_2}{y_2}\cdots\int_0^{y_{n-1}}\frac{dy_n}{y_n-1} = -G(o,o,..,o,1;x)$

We are trying to do

$$\int_0^1 \prod_i d\xi_i \frac{1}{P(\xi_i,p_j)} = \sum\prod G(\vec{b}_m,x)$$

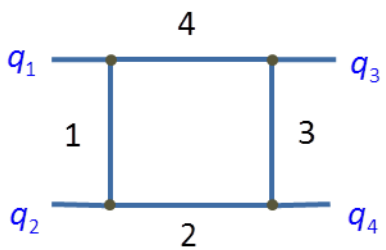
Separation of IR/UV

Summary



Example

Massless one loop box diagram



$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$
$$\mathcal{V} = t\,\alpha_1\alpha_3 + s\,\alpha_2\alpha_4$$

$$F(s,t;\varepsilon) = \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_4 \delta\left(\sum \alpha_j - 1\right) \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)^{2\varepsilon}}{\left(-t\,\alpha_1\alpha_3 - s\alpha_2\alpha_4\right)^{2+\varepsilon}}$$

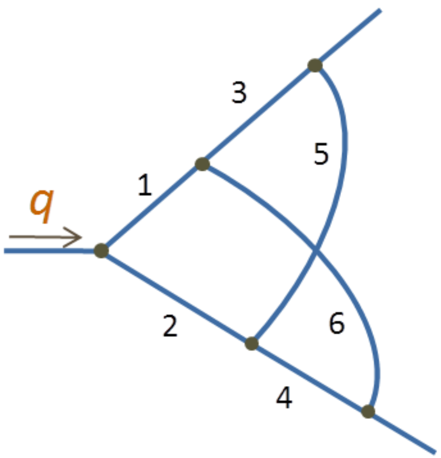
After IR separation

$$F(s,t;\varepsilon) = \frac{(-t)^{-\varepsilon}}{st} \Gamma(2+\varepsilon) \frac{\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)} \int_0^1 \xi_1\xi_2\ (\xi_2\bar{\xi}_2)^{-1-\varepsilon} \left(x\xi_1\xi_2 + \bar{\xi}_2\bar{\xi}_2\right)^{\varepsilon}$$

The result is

$$F(s,t;\varepsilon) = \frac{1}{st}\left(\frac{4}{\varepsilon^2} - \frac{2}{\varepsilon}\left(\ln(-s)+\ln(-t)\right) + 2\ln(-s)\ln(-t) - \frac{4\pi^2}{3}\right)$$

Example



$$\mathcal{U} = (\alpha_3+\alpha_5)(\alpha_4+\alpha_6)+(\alpha_1+\alpha_2)(\alpha_3+\alpha_5+\alpha_4+\alpha_6)$$

$$\mathcal{V} = q^2(\alpha_1\alpha_2(\alpha_3+\alpha_4+\alpha_5+\alpha_6)+\alpha_1\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_6)$$

After IR separation

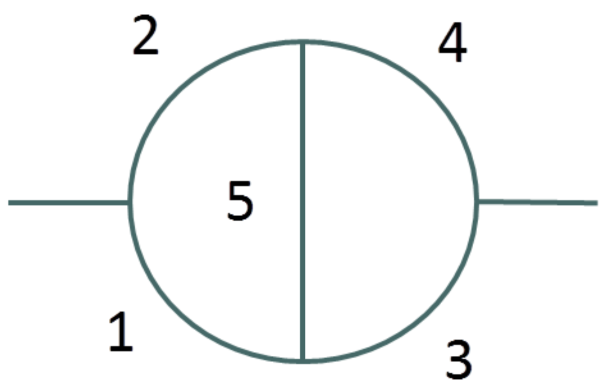
$$F(q^2;\varepsilon) = \frac{\Gamma(2+2\varepsilon)}{(-q^2)^{2+2\varepsilon}}\int_0^1 \xi_1\xi_2\xi_3\xi_4\xi_5\ (\xi_1\bar{\xi}_1\xi_2\bar{\xi}_2)^{-1-2\varepsilon}(\xi_3\bar{\xi}_3)^{-1-\varepsilon}\,\xi_4^{-1-2\varepsilon}\,\bar{\xi}_4^{1+\varepsilon}\,\xi_5^{-\varepsilon}$$

$$\times\left(\xi_4\bar{\xi}_5+\bar{\xi}_4\right)^{-2-2\varepsilon}(\xi_5\bar{\xi}_4+\xi_1\xi_2\xi_4\xi_5+\bar{\xi}_1\bar{\xi}_2\xi_4\bar{\xi}_5)^{3\varepsilon}$$

$$= \frac{1}{(-q^2)^{2+2\epsilon}}\left(\frac{1}{\epsilon^4}-\frac{\pi^2}{\epsilon^2}-\frac{83\zeta(3)}{3\epsilon}-\frac{59\pi^4}{120}\right)$$

Example

Massless two loop diagram



$$\mathcal{U} = \alpha_{1234}\alpha_5 + \alpha_{12}\alpha_{34}$$
$$\mathcal{V} = q^2\left(\alpha_5\alpha_{13}\alpha_{24} + \alpha_1\alpha_3\alpha_{24} + \alpha_2\alpha_4\alpha_{13}\right)$$

There is no div. term.

$$F(s,t;\varepsilon) = \frac{1}{q^2}\int_0^1 d\xi_1 \cdots \int_0^1 d\xi_4 \frac{1}{(\bar{\xi}_2\xi_2\xi_3 + \xi_1(1 - \bar{\xi}_2\xi_3(1 + \xi_2 - 2\xi_4) - \xi_4 - \bar{\xi}_2{}^2\xi_3^2\xi_4))}$$

$$F(s,t;\varepsilon) = \frac{6}{q^2}\zeta(3)$$

Summary

- We propose an analytic method of extracting IR/UV divergences in multiloop diagrams representing the results in terms of GHPLs.
- This method is algorithmic and can be automatized with computer programing
- We are testing massless 2loop 4point function as well as some massive diagram. We hope to soon make it public.
- This method will be useful for higher order corrections in this precision era.